

# On equidistributing principles in moving finite element methods

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**Abstract:** This paper examines approximate equidistributing principles derived by Herbst, Schoombie and Michell for two moving finite element methods applied to a simple transport equation and shows that these principles are too weak to distinguish between alternative node distributions. Stronger distributing principles are derived that determine the asymptotic node distribution uniquely, provided the initial distribution is given.

**Keywords:** Moving finite element method, convection-diffusion, equidistributing principle.

## 1. Introduction

We are writing this paper in response to a paper by Herbst, Schoombie and Mitchell [2]. In an earlier paper, Herbst et al. [1] introduced a moving Petrov–Galerkin method for the solution of transport equations. This method uses Petrov–Galerkin variational equalities to govern both the evolution of the approximate solution values and the motion of the system of grid points. Miller and Miller [3,4] had previously introduced an analogous moving finite element method based on Galerkin variational equalities. The Petrov–Galerkin method was a true advance as it yielded a superior distribution of grid points compared to the Galerkin method.

In the follow-up paper, Herbst et al. [2] attempted to analyze the node distributions obtained by the two methods, and derived the equations

$$h_{i+1}u_{xx}(x_{i+}, t) = h_i u_{xx}(x_{i-}, t) + O(h^2), \quad \text{and} \quad (1)$$

$$h_i^2 u_{xx}(x_{i-}, t) = h_{i+1}^2 u_{xx}(x_{i+}, t) + O(h^3), \quad (2)$$

where  $x_0, x_1, \dots, x_N$  are node points,  $h_i = x_i - x_{i-1}$ , and  $h = \max h_i$ . These equations are described in [2] as approximate equidistributing principles that describe the distribution of nodes obtained by the Galerkin and Petrov–Galerkin methods, respectively. In our view (1) and (2) are too weak to be called equidistributing principles. As we show, additional information about the  $O(h^2)$  and  $O(h^3)$  terms is needed to determine the asymptotic node distribution obtained.

Indeed, (1) and (2) are satisfied by a wide variety of node distributions, including the uniform distribution. We also derive, under modest assumptions regarding the smoothness of the exact solution and the rate of convergence of the approximate solution, stronger distributing principles in the form of partial differential equations that uniquely determine the asymptotic node distributions given appropriate initial and boundary conditions.

## 2. Background

We will briefly describe the moving finite element methods discussed in [1,2]. We adopt the same notation. Consider the transport equation

$$u_t = \epsilon u_{xx} - V(u)_x, \quad (3)$$

and let  $v(x, t)$  be an approximate solution to this equation together with associated initial and boundary conditions that we will leave unspecified. Assume that  $v(x, t)$  is continuous and piecewise linear, and can be written as

$$v(x, t) = v(x; x_0, \dots, x_N, a_0, \dots, a_N), \quad (4)$$

where  $x_i = x_i(t)$  are the nodes and  $a_i = a_i(t) = v(x_i, t)$  are the nodal values. Then

$$v_t = \sum_{i=0}^N \left( \dot{a}_i \frac{\partial v}{\partial a_i} + \dot{x}_i \frac{\partial v}{\partial x_i} \right) = \sum_{i=0}^N (\dot{a}_i \alpha_i + \dot{x}_i \beta_i). \quad (5)$$

The functions  $\alpha_i$  are hat functions,

$$\alpha_i(x) = \frac{\partial v}{\partial a_i} = \begin{cases} (x - x_{i-1})/h_i, & x_{i-1} \leq x \leq x_i, \\ (x_{i+1} - x)/h_{i+1}, & x_i \leq x \leq x_{i+1}, \\ 0, & \text{otherwise;} \end{cases} \quad (6)$$

the functions  $\beta_i$  are modified hat functions with a potential discontinuity at  $x_i$ ,

$$\beta_i(x) = \frac{\partial v}{\partial x_i} = -v_x \alpha_i = \begin{cases} -m_i(x - x_{i-1})/h_i, & x_{i-1} \leq x \leq x_i, \\ -m_{i+1}(x_{i+1} - x)/h_{i+1}, & x_i \leq x \leq x_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \quad (7)$$

where  $m_i = (a_i - a_{i-1})/h_i$  is the slope of  $v$  on the  $i$ th element.

Miller and Miller [3,4], starting from a least squares principle, developed a semi-discrete Galerkin moving point method in which at any time the derivatives  $\dot{a}_i$  and  $\dot{x}_i$  are determined by the equations

$$\begin{aligned} \sum_{j=0}^N [(\alpha_i, \alpha_j) \dot{a}_j + (\alpha_i, \beta_j) \dot{x}_j] &= (\alpha_i, \epsilon v_{xx} - V(v)_x), \\ \sum_{j=0}^N [(\beta_i, \alpha_j) \dot{a}_j + (\beta_i, \beta_j) \dot{x}_j] &= (\beta_i, \epsilon v_{xx} - V(v)_x), \end{aligned} \quad (8)$$

where  $(f, g)$  denotes the usual inner product  $\int_a^b fg \, dx$ , and  $[a, b]$  is the problem interval. From

these equations, Herbst et al. [2] derived

$$h_{i+1}[\dot{v}_{i+1-} + 2\dot{v}_{i+}] - h_i[2\dot{v}_{i-} + \dot{v}_{i-1+}] + 6 \int_0^1 [V(a_i + m_{i+1}h_{i+1}\tau) + V(a_i - m_i h_i \tau)] d\tau - 12V(a_i) = 0, \quad (9)$$

from which in turn (1) was derived. Here  $\dot{v}_{i\pm}$  are the right and left limits of  $v_i$  at  $x_i$ .

In [1] Herbst et al. introduced a Petrov–Galerkin method analogous to the method of Miller and Miller except that hermite cubics are used as test functions. One basis for the Hermite cubics consists of the functions

$$S_i(x) = \alpha_i(x)^2[3 - 2\alpha_i(x)], \quad \text{and} \quad (10)$$

$$T_i(x) = \alpha_i(x)^2[\alpha_i(x) - 1]/[d\alpha_i(x)/dx].$$

Thus, the time derivatives  $\dot{a}_i$  and  $\dot{x}_i$  are determined by the equations

$$\sum_{j=0}^N [(S_i, \alpha_j)\dot{a}_j + (S_i, \beta_j)\dot{x}_j] = (S_i, \epsilon v_{xx} - V(v)_x), \quad (11)$$

$$\sum_{j=0}^N [(T_i, \alpha_j)\dot{a}_j + (T_i, \beta_j)\dot{x}_j] = (T_i, \epsilon v_{xx} - V(v)_x).$$

From these equations, Herbst et al. [1,2] derived

$$h_{i+1}^2[2\dot{v}_{i+1-} + 3\dot{v}_{i+}] - h_i^2[3\dot{v}_{i-} + 2\dot{v}_{i-1+}] - 60 \int_0^1 [h_i V(a_i - m_i h_i \tau) + h_{i+1} V(a_i + m_{i+1} h_{i+1} \tau)] (1 - \tau)(1 - 3\tau) d\tau = 0, \quad (12)$$

from which (2) was derived.

### 3. Equidistributing principles

In this section we will show that, loosely speaking, (1) and (2) are satisfied by any smooth distribution of nodes. For the sake of comparison, we will then examine the approximate equidistributing principle introduced by Pereyra and Sewell [5]. We will make use of the concept of a grading function, which we define to be an increasing function  $g$  from the interval  $[0, 1]$  into the problem interval  $[a, b]$  such that  $g(0) = a$  and  $g(1) = b$ . Grading functions provide a convenient means for generating a grid  $\pi = (x_0, \dots, x_N)$  on  $[a, b]$  as the image of a uniform grid on  $[0, 1]$ , that is, by letting  $x_i = g(i/N)$ . In particular, the uniform grid on  $[a, b]$  is generated by  $g(\xi) = a + (b - a)\xi$ . Carey and Dinh [6] have used grading functions to advantage in the construction of grids for polynomial interpolation and the solution of two-point boundary value problems.

We will also need the concept of an asymptotic grading function. Consider a family of grids  $\Pi = \{\pi^N\}$  where each  $\pi^N$  is the grid  $a = x_0^N < x_1^N < \dots < x_N^N = b$ . We will usually delete the superscript  $N$ . A grading function  $g$  is an asymptotic grading function for  $\Pi$  if for any sequence of indices  $i(N)$  for which  $x_{i(N)}$  converges we have

$$\lim_{N \rightarrow \infty} x_{i(N)} = \lim_{N \rightarrow \infty} g(i(N)/N).$$

We now state an assumption we will need below:

(A) There exists a grading function  $g$  with a continuous second derivative on  $[0, 1]$  for which

$$h_i = g(i/N) - g((i-1)/N) + O(1/N^2).$$

Note that under this assumption  $x_i = g(i/N) + O(1/N)$  so that  $g$  is an asymptotic grading function for  $\Pi$ .

We now show that (1) and (2) are satisfied under very general conditions.

**Lemma 1.** Suppose  $u(x)$  has a continuous second derivative and  $\Pi$  is a family of grids for which assumption (A) is satisfied. Then (1) and (2) are also satisfied.

**Proof.** Since  $g$  has a continuous second derivative, we have

$$\begin{aligned} h_i &= g(i/N) - g((i-1)/N) + O(1/N^2) = g'(i/N)/N + O(1/N^2) \\ &= g'(i/N)/N + O(h^2). \end{aligned}$$

Similarly  $h_{i+1} = g'(i/N)/N + O(h^2)$ . Thus

$$h_{i+1} - h_i = O(h^2). \quad (13)$$

Multiplying (13) by  $u_{xx}(x_i) = u_{xx}(x_{i-}) = u_{xx}(x_{i+})$  we get (1). Also multiplying (13) by  $(h_{i+1} + h_i)u_{xx}(x_i)$  gives

$$h_{i+1}^2 u_{xx}(x_i) - h_i^2 u_{xx}(x_i) = O(h^3), \quad (14)$$

which is equivalent to (2).  $\square$

Equations (1) and (2) are actually weaker than Lemma 1 indicates since  $h$  may be of lower order than  $1/N$ ; in fact,  $h$  need not converge to zero. For example, the family of grids with one point at 0 and  $N$  points distributed evenly from  $\frac{1}{2}$  to 1 has  $h = \frac{1}{2}$  for all  $N$  and satisfies both (1) and (2).

The term ‘approximately equidistributing’ was first introduced by Pereyra and Sewell [5] to apply to meshes which satisfied equations such as

$$h_i^{np+1} f(x_i)^p = K(1/N)^{np+1} [1 + O(h)], \quad (15)$$

where  $f$  is a function on  $[a, b]$  related to local truncation error,  $K$  is a constant related to  $f$ ,  $n$  is the order of local truncation error, it is desired to minimize error in the  $L^p$  norm, and the function  $f$  is constructed to insure that  $h$  is of order  $1/N$ . Equation (15), together with assumption (A) and the assumption that  $f$  is continuous on  $[a, b]$ , specifies the asymptotic grading function uniquely. In fact, under these assumptions we get

$$\xi = g^{-1}(x) = \int_a^x f(x)^\sigma dx / \int_a^b f(x)^\sigma dx, \quad (16)$$

where  $\sigma = p/(np + 1)$ .

Of course, the reason that (1) and (2) are weaker than (15), is because they relate each  $h_i$  only to its immediate neighbors  $h_{i-1}$  and  $h_{i+1}$ . If these equations are summed so as to relate distant intervals, the  $O(h^2)$  and  $O(h^3)$  terms may become  $O(h)$  and  $O(h^2)$  terms, respectively, and the summed equations are then meaningless. This difficulty is overcome in the next section by treating explicitly those terms that contribute significantly to the summation.

#### 4. Moving finite element meshes

We will now build on (9) and (12) of Herbst et al. to derive distributing principles comparable in strength to (15). Let  $u(x, t)$  be the exact solution to (3) and its associated initial and boundary conditions. We will again need a grading function, but now  $g$  will also be a function of time and will define a transformation  $[0, 1] \times [0, T] \rightarrow [a, b] \times [0, T]$  as follows:  $x = g(\xi, \tau)$ ,  $t = \tau$ .

Assumption (A) will need to be revised to accommodate time. In addition, in many problems there will be points with a zero asymptotic node density. At these points  $x_\xi = \partial g / \partial \xi$  will be infinite. Thus our first assumption regarding the asymptotic distribution of nodes will be as follows:

(I) The function  $g(\xi, \tau)$  is a continuous grading function. Moreover,  $g$  has continuous third derivatives except possibly at a finite number of exceptional points (whose number and location may vary in time).

Now let  $[A, B]$  be an arbitrary subinterval of  $[a, b]$  that includes no exceptional points. The end points  $A$  and  $B$  may vary with time. Define  $j = j(N)$  and  $k = k(N)$  so that  $x_{j-1}$  is the first node of  $\pi^N$  in  $[A, B]$  and  $x_{k+1}$  is the last. Redefine  $h$  to be  $\max\{h_i | j \leq i \leq k+1\}$ . The notation  $\alpha = O(\beta)$  will be understood to mean  $|\alpha| \leq K\beta$  for  $N$  sufficiently large, where  $K$  may depend only on  $t$ ,  $A$  and  $B$  and the inequality applies uniformly over any other variables upon which the expressions  $\alpha$  and  $\beta$  may depend. We may now state the further assumption:

(II) The function  $g$  is an asymptotic grading function for  $II$ . In addition, for any subinterval  $[A, B]$  of  $[a, b]$  that includes no exceptional points we have

$$x_i = g(i/N) + O(1/N) \quad \text{and} \quad \dot{x}_i = g_\tau(i/N) + O(1/N),$$

for  $j-1 \leq i \leq k+1$ , and

$$x_i - x_{i-1} = g(i/N) - g((i-1)/N) + O(1/N^2) \quad \text{and}$$

$$\dot{x}_i - \dot{x}_{i-1} = g_\tau(i/N) - g_\tau((i-1)/N) + O(1/N^2),$$

for  $j \leq i \leq k+1$ .

We also need to make some assumptions regarding the smoothness of  $V$  and  $u$  and the rate of convergence of the approximate solution:

(iii) The functions  $V(u)$  and  $u(x, t)$  have continuous third derivatives.

(iv) For any subinterval  $[A, B]$  of  $[a, b]$  that includes no exceptional points, the solution values  $a_i$  satisfy

$$a_i = u(x_i) + O(1/N) \quad \text{and} \quad \dot{a}_i = u_t(x_i) + u_x(x_i)\dot{x}_i + O(1/N),$$

for  $j-1 \leq i \leq k+1$  and

$$a_i - a_{i-1} = u(x_i) - u(x_{i-1}) + O(1/N^2) \quad \text{and}$$

$$\dot{a}_i - \dot{a}_{i-1} = u_t(x_i) - u_t(x_{i-1}) + u_x(x_i)\dot{x}_i - u_x(x_{i-1})\dot{x}_{i-1} + O(1/N^2),$$

for  $j \leq i \leq k+1$ .

These assumptions yield some corollaries that we will need. The second equation of (II) and the continuity of  $x_\tau = \partial g / \partial \tau$  show that  $\dot{x}_i$  is bounded, while the third and fourth equations of (II) and the continuity of  $x_{\xi\xi}$  and  $x_{\xi\xi\tau}$  show that

$$h_i = (1/N)x_\xi(i/N) + O(1/N^2) = O(1/N) \quad \text{and}$$

$$\dot{h}_i = (1/N)x_{\xi\tau}(1/N) + O(1/N^2) = O(1/N). \quad (17)$$

In particular,  $h = O(1/N)$ , so that  $O(h)$  and  $O(1/N)$  are interchangeable. Further, the third equation of (IV) and continuity of  $u_{xx}$  give

$$m_i = u_x(x_i) + O(h) = u_x(x_{i-1}) + O(h). \quad (18)$$

It also follows that  $m_i$  is bounded. Assumptions (III) and (IV) also imply

$$\begin{aligned} \dot{v}_{i-} &= \dot{a}_i - m_i \dot{x}_i = u_t(x_i) + [u_x(x_i) - m_i] \dot{x}_i + O(h) \\ &= u_t(x_i) + \dot{x}_i O(h) + O(h) \\ &= u_t(x_i) + O(h) \quad \text{and} \\ \dot{v}_{i+} &= \dot{a}_i - m_{i+1} \dot{x}_i = u_t(x_i) + O(h). \end{aligned} \quad (19)$$

Moreover,

$$\begin{aligned} \dot{m}_i &= [\dot{a}_i - \dot{a}_{i-1} - m_i(\dot{x}_i - \dot{x}_{i-1})]/h_i \\ &= \{u_t(x_i) - u_t(x_{i-1}) + [u_x(x_i) - u_x(x_{i-1})] \dot{x}_i + [u_x(x_{i-1}) - m_i] \dot{h}_i + O(h^2)\}/h_i \\ &= u_{xt}(x_i) + u_{xx}(x_i) \dot{x}_i + O(h) = u_{xt}(x_i) + u_{xx}(x_i) x_\tau(i/N) + O(h) \\ &= u_{x\tau}(x_i) + O(h). \end{aligned} \quad (20)$$

We will also need to apply Taylor series expansions to  $V(a_i - m_i h_i \tau)$  and similar expressions. We note that the boundedness of  $m_i$  and  $\tau \in [0, 1]$  permits us to express the remainder as a power of  $O(h)$ .

We first analyze the asymptotic node distribution for the method of Herbst et al. Expanding  $V$  in a Taylor series about  $a_i$  in (12), performing the indicated integrations, and summing from  $i=j$  to  $i=k$ , we get

$$\begin{aligned} 0 &= h_{k+1}^2(3\dot{v}_{k+1-} + 2\dot{v}_{k+}) - h_j^2(2\dot{v}_{j-} + 3\dot{v}_{j-1+}) + 5m_{k+1}h_{k+1}^2V'(a_k) - 5m_jh_j^2V'(a_{j-1}) \\ &\quad - \sum_{i=j}^{k+1} h_i^2(\dot{v}_{i-} - \dot{v}_{i-1+}) - \sum_{i=j}^k 5m_ih_i^2[V'(a_i) - V'(a_{i-1})] \\ &\quad + \sum_{i=j}^k 2m_i^2h_i^3[V''(a_i) + V''(a_{i-1})] + O(h^3). \end{aligned} \quad (21)$$

Recognizing that  $\dot{v}_{i-} - \dot{v}_{i-1+} = \dot{m}_i h_i$  and expanding  $V'(a_{i-1})$  and  $V''(a_{i-1})$  in Taylor series about  $a_i$ , (21) becomes

$$\begin{aligned} 0 &= h_{k+1}^2(3\dot{v}_{k+1-} + 2\dot{v}_{k+}) - h_j^2(2\dot{v}_{j-} + 3\dot{v}_{j-1+}) \\ &\quad + 5m_{k+1}h_{k+1}^2V'(a_k) - 5m_jh_j^2V'(a_{j-1}) - \sum_{i=j}^{k+1} \dot{m}_i h_i^3 - \sum_{i=j}^k m_i^2 h_i^3 V''(a_i) + O(h^3). \end{aligned} \quad (22)$$

Next, invoking convergence results (18)–(20), applying the chain rule  $u_x V' = V_x$ , and using (3), (22) simplifies to

$$\begin{aligned} 0 &= 5\epsilon [h_{k+1}^2 u_{xx}(x_k) - h_j^2 u_{xx}(x_j)] \\ &\quad - \sum_{i=j}^{k+1} h_i^3 u_{x\tau}(x_i) - \sum_{i=j}^k h_i^3 u_x(x_i)^2 V''(u(x_i)) + O(h^3). \end{aligned} \quad (23)$$

Equation (23) is the stronger version of (2) that we sought, but it is awkward. Thus we invoke equation (17) which permits us to express this result in terms of the asymptotic grading function. Multiplying (23) by  $N^2$ , applying (17), and letting  $N \rightarrow \infty$ , we get

$$5\epsilon \left[ u_{xx}/\xi_x^2 \right]_A^B = \int_A^B \frac{u_{x\tau} + u_x^2 V''}{\xi_x^2} dx. \quad (24)$$

This equation, after differentiation and making the substitution

$$u_{x\tau} + V''u_x^2 = \epsilon u_{xxx} + u_{xx}(x_\tau - V'),$$

can be solved for the asymptotic node velocity  $x_\tau$  to get

$$x_\tau = V' + 2\epsilon [2u_{xxx}/u_{xx} - 5\xi_{xx}/\xi_x]. \quad (25)$$

We can also integrate this equation to get an expression for the asymptotic node density  $\xi_x$ ,

$$\xi_x = K |u_{xx}|^{2/5} \exp \left\{ \frac{1}{10\epsilon} \int V' - x_\tau dx \right\}. \quad (26)$$

In the steady-state case,  $x_\tau = 0$  and  $V' = \epsilon u_{xx}/u_x$ , so we get

$$\xi_x = K |u_{xx}|^{2/5} |u_x|^{1/10}. \quad (27)$$

The same derivation, beginning with (9) produces similar results for the method of Miller and Miller. The principle results are

$$0 = 3\epsilon \left[ h_{k+1} u_{xx}(x_k) - h_j u_{xx}(x_j) \right] - \sum_{i=j}^{k+1} h_i^2 u_{x\tau}(x_i) - \sum_{i=j}^k h_i^2 u_x(x_i)^2 V''(u(x_i)) + O(h^2), \quad (28)$$

$$x_\tau = V' + \epsilon [2u_{xxx}/u_{xx} - 3\xi_{xx}/\xi_x], \quad (29)$$

$$\xi_x = K |u_{xx}|^{2/3} \exp \left\{ \frac{1}{3\epsilon} \int V' - x_\tau dx \right\}, \quad (30)$$

and in the steady-state case,

$$\xi_x = K |u_{xx}|^{2/3} |u_x|^{1/3}. \quad (31)$$

## 5. Numerical results

In order to verify the above equations, specifically (30) and (31), we implemented the moving finite element method of Miller and Miller (without using a penalty function) and applied it to two test problems, one steady-state and one transient. We describe the steady-state problem first.

We examine the problem

$$\begin{aligned} -(2\sqrt{u})_x + u_{xx} &= 0 \quad \text{for } 0 < x < 1, \\ u(0) &= 0 \quad \text{and} \quad u(1) = 1. \end{aligned} \quad (32)$$

The solution is  $u(x) = x^2$ . Equation (31) gives the asymptotic grading function for this problem as  $x = g(\xi) = \xi^{3/4}$ . This function is shown in Fig. 1. We choose this problem because any

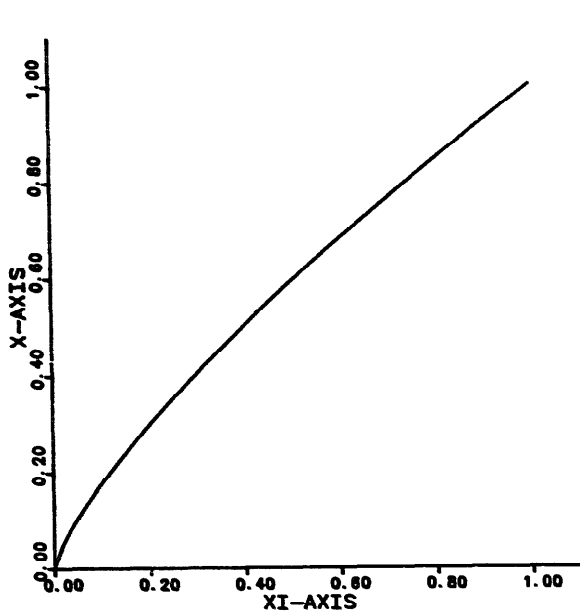


Fig. 1. Asymptotic grading function for the steady-state problem.

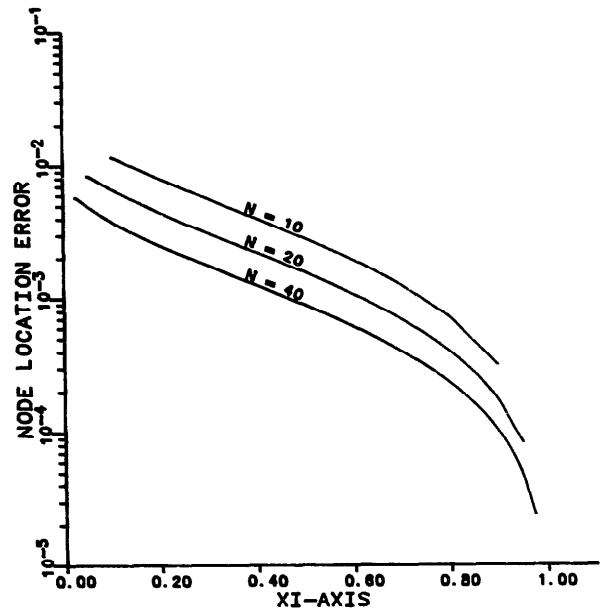


Fig. 2. Node location errors for the steady-state problem.

equidistribution principle based solely on  $u_{xx}$ , such as (1), would predict a uniform distribution of nodes since  $u_{xx}$  is constant. We made three runs of the Miller and Miller method applied to this problem, the first with  $N = 10$  elements, the second with  $N = 20$  elements and the third with  $N = 40$  elements. We found that the distribution of nodes does approach the distribution predicted by (31). The convergence is approximately first order. This result is displayed in Fig. 2, which shows the node location error NLE of node  $x_i$ , defined to be

$$\text{NLE} = |x_i - g(i/N)|, \quad (33)$$

plotted against  $\xi = i/N$  for all three runs. We note that the terminology 'node location error' is justified if we regard  $g(i/N)$  as a predictor of the actual node location  $x_i$ .

For the transient case we applied the Miller and Miller method to the convection-diffusion problem

$$\begin{aligned} u_t &= -u_x + \epsilon u_{xx} & \text{for } 0 < x < 1 \text{ and } t > 0, \\ u &= 0 & \text{for } 0 < x < 1 \text{ and } t = 0, \\ u &= 1 & \text{for } x = 0 \text{ and } t \geq 0, \\ u &= 0 & \text{for } x = 1 \text{ and } t \geq 0, \end{aligned} \quad (34)$$

with  $\epsilon = 0.0016$  for a Peclet number of 625. We examined the solution obtained at the time  $t = 0.5$ . Since the solution front has not reached the downstream boundary at this time, the well known analytic solution for the semi-infinite case,

$$u(x, t) = \text{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) + \exp\left(\frac{x}{\epsilon}\right) \text{erfc}\left(\frac{x+t}{2\sqrt{\epsilon t}}\right), \quad (35)$$

may be used for the exact solution.



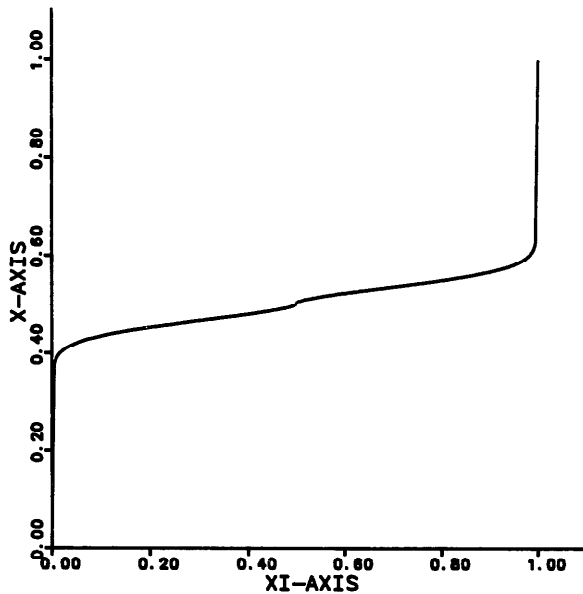


Fig. 3. Asymptotic grading function for the transient problem at time  $t = 0.5$ .

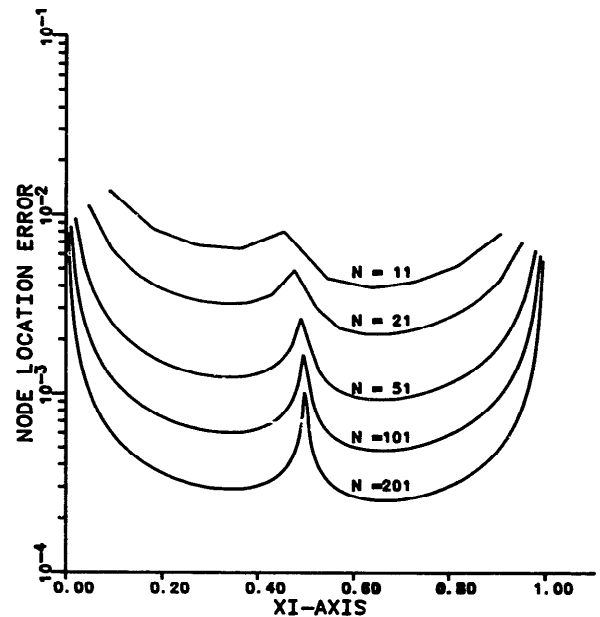


Fig. 4. Node location errors for the transient problem at time  $t = 0.5$ .

We faced two difficulties in constructing the asymptotic grading function for this problem from (30). First, the solution contains an inflection point in the center of the moving front. This inflection point is exceptional under assumption (I) so that (30) does not apply there. In particular, the constant of integration  $K$  may be different on one side of the inflection point than on the other. We avoided this difficulty by choosing the problem and the time  $t = 0.5$  so that the solution is symmetrical about the inflection point. In this case  $K$  is the same on both sides. The second difficulty is that we do not know the asymptotic node velocity  $x_\tau = \partial g / \partial \tau$ . We approximated  $x_\tau$  by  $\dot{x}_i$  at  $x_i$  and interpolated linearly elsewhere. Judging from the results below, this approximation was adequate. The asymptotic grading function that we obtained is shown in Fig. 3.

We made five runs of the Miller and Miller algorithm applied to the transient problem, with  $N = 11, 21, 51, 101$  and  $201$  elements. We used odd numbers of elements to avoid placing a node at the inflection point. Figure 4 shows the node location errors obtained in these runs. Again, the actual node distributions approach the predicted asymptotic distribution with approximately first order convergence.

## References

- [1] B.M. Herbst, S.W. Schoombie and A.R. Mitchell, A moving Petrov-Galerkin method for transport equations, *Internat. J. Numer. Meths. Engrg.* **18** (1982) 1321-1336.
- [2] B.M. Herbst, S.W. Schoombie and A.R. Mitchell, Equidistributing principles in moving finite element methods, *J. Comput. Applied Math.* **9** (1983) 377-389.

- [3] K. Miller, Moving finite element methods, part II, *SIAM J. Numer. Anal.* **18** (1981) 1033–1057.
- [4] K. Miller and R.N. Miller, Moving finite element methods, part I, *SIAM J. Numer. Anal.* **18** (1981) 1019–1032.
- [5] V. Pereyra and E.G. Sewell, Mesh selection for discrete solution of boundary problems in ordinary differential equations, *Numer. Math.* **23** (1975) 261–268.
- [6] G.F. Carey and H.T. Dinh, Grading functions and mesh redistribution, *SIAM J. Numer. Anal.* **22** (1985) 1028–1040.